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AN INTEGRAL-INTERPOLATORY ITERATIVE METHOD FOR THE SOLUTION OF NON-LINEAR SCALAR EQUATIONS

Boleslaw Kacewicz

Carnegie-Mellon University

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This paper deals with the iterative solution of non-linear equations $f(x) = 0$. We consider integral information on f which is given by $f(x_0), f'(x_0), \ldots, f^{(s)}(x_0)$ and $\int_{y_0}^{x_0} f(t)dt$. We define an interpolatory-integral method which uses integral information and which has maximal order of convergence equal to $s+3$. Since the maximal order of iterations which use $f(x_0), \ldots, f^{(s)}(x_0)$ is equal to $s+1$, the additional information given by the integral $\int_{y_0}^{x_0} f(t)dt$ increases the order by two.

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AN INTEGRAL - INTERPOLATORY ITERATIVE METHOD FOR THE SOLUTION OF NON-LINEAR SCALAR EQUATIONS

Boleslaw Kacewicz
Department of Computer Science
Carnegie-Mellon University
Pittsburgh, PA. 15213
(Visiting from the University of Warsaw)
January 1975

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ABSTRACT

This paper deals with the iterative solution of non-linear equations $f(x) = 0$. We consider integral information on f which is given by $f(x_0), f'(x_0), \ldots, f^{(s)}(x_0)$ and $\int_{y_0}^{x_0} f(t)dt$. We define an interpolatory-integral method which uses integral information and which has maximal order of convergence equal to s+3. Since the maximal order of iterations which use $f(x_0), \ldots, f^{(s)}(x_0)$ is equal to s+1, the additional information given by the integral $\int_{y_0}^{x_0} f(t)dt$ increases the order by two.

1. INTRODUCTION

We consider the solution of the nonlinear scalar equation

$$(1.1)\quad f(x) = 0,$$

where f is a complex function of complex variable.

In most papers which deal with stationary iterative methods for (1.1) it is assumed we know the standard information for f (Wozniakowski [74])

$$\mathfrak{N}_s = \{f(x_0), \ldots, f^{(s)}(x_0)\}$$

where $s \geq 1$ and x_0 is an approximation to the solution α. The maximal order of convergence of such methods is equal to s + 1 (Traub [64], Wozniakowski [73]). We raise the question how other types of information can be used in iterative processes and what is the maximal order of convergence for this information.

This paper deals with integral information which consists of the standard information $\mathfrak{N}_s$ and additionally the value of an integral. Thus

$$(1.2)\quad \mathfrak{N}_{-1,s} = \{f(x_0), \ldots, f^{(s)}(x_0), \int_{y_0}^{x_0} f(t)\,dt\}$$

where y_0 is a complex number defined in Section 3.

In Section 2 we define an interpolatory - integral method $I_{-1,s}$ which uses integral information $\mathfrak{N}_{-1,s}$ to estimate α and in Section 7 we prove its order for $s \geq 1$ is maximal. Sections 4, 5 and 6 contain theorems about the convergence of $I_{-1,s}$.

Wozniakowski [74] defined for the generalized information $\mathfrak{N}$ an order of information $p(\mathfrak{N})$ and proved it is equal to the maximal order of convergence. In Section 7 we prove that for $s \geq 1$ and for suitable chosen y_0, $p(\mathfrak{N}_{-1,s}) = s + 3$. Since $p(\mathfrak{N}_s) = s + 1$, the additional information given by $\int_{y_0}^{x_0} f(t)dt$ increases the order of information by two. For systems of nonlinear equations similar results can be proved and will be reported to a future paper.

2. INTERPOLATORY - INTEGRAL ITERATIVE METHOD $I_{-1,s}$

Let us consider the solution of the nonlinear equation,

(2.1) $f(x) = 0,$

where $f: D \to \mathbb{C}$, D is an open subset of $\mathbb{C}$, $\mathbb{C}$ denotes the set of complex numbers. Let $\alpha \in D$ be a simple zero of f, $f(\alpha) = 0 \neq f'(\alpha)$. An interpolatory - integral method $I_{-1,s}$ is defined as follows. Let x_i be an approximation of α. We assume that the information on f is given by

(2.2) $\mathfrak{N}_{-1,s} = \mathfrak{N}_{-1,s}(x_i;f) = \{f(x_i),\ldots,f^{(s)}(x_i), \int_{y_i}^{x_i} f(t)\,dt\},$

where y_i depends on x_i, $f^{(k)}(x_i)$, $k = 0,1,\ldots,s$, $y_i \neq x_i$ and is defined in Section 3. If $s = 0$ then y_i can depend on x_i, $f(x_i)$, x_{i-1}, $f(x_{i-1})$. The value of y_i will be chosen to maximize the order of iteration.

The information consists of the standard information given by $f(x_i),\ldots,f^{(s)}(x_i)$ and additionally the value of the integral. Next, let w_i be an interpolatory polynomial of degree at most $s+1$ such that:

(2.3) $w_i^{(k)}(x_i) = f^{(k)}(x_i) \qquad k = 0,1,\ldots,s,$

(2.4) $\int_{y_i}^{x_i} w_i(t)\,dt = \int_{y_i}^{x_i} f(t)\,dt\,.$

If w_i exists then the next approximation x_{i+1} in $I_{-1,s}$ method is defined as a zero of polynomial w_i,

(2.5) $w_i(x_{i+1}) = 0,$

with a criterion to make x_{i+1} unique. We shall now prove that w_i exists and is unique. Let

(2.6) $F(x) = \int_{y_i}^{x} f(t)\,dt,$

and let g_i be a polynomial of degree $\le s+2$ such that

(2.7) $g_i(y_i) = F(y_i) = 0,$

(2.8) $g_i^{(k)}(x_i) = F^{(k)}(x_i), \qquad k = 0,1,\ldots,s+1.$

Thus, g_i is a Hermite interpolatory polynomial for F. The assumption $y_i \neq x_i$ implies the existence and the uniqueness of g. Set

(2.9) $w_i(x) = g_i'(x).$

Then (2.3) and (2.4) are satisfied which completes the proof. Moreover, from (2.8) and (2.9) follows an error formula,

$$(2.10)\quad F(x) - g_i(x) = (x - y_i)(x - x_i)^{s+2}\, G_i(x)$$

where

$$(2.11)\quad G_i(x) = G_i(x,f) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{s+2}} f^{(s+2)}(y_i + t_1(x_i - y_i) + t_{s+3}\cdot$$

$$(x - x_i))\, dt_1 \ldots dt_{s+3}.$$

Differentiating (2.10) we get from (2.11),

$$(2.12)\quad f(x) - w_i(x) = R(x)$$

where

$$R(x) = R(x,f) = (x - x_i)^{s+1}\{[(s+2)(x - y_i) + x - x_i]\, G_i(x) + G_i'(x)(x - x_i)(x - y_i)\}.$$

3. DEFINITION OF A LOWER LIMIT OF THE INTEGRAL

We want to define y_i to maximize the order of $I_{-1,s}$. Setting $x = \alpha$ in (2.12) we have

$$(3.1)\quad -w_i(\alpha) = R(\alpha)$$

Let us assume for a moment that w_i has a zero x_{i+1} sufficiently close to a simple zero α.

$$x_{i+1} - \alpha = -\frac{w_i(\alpha)}{w_i'(\alpha)} + O((x_{i+1} - \alpha)^2) = O(R(\alpha))$$

We see that the order of iteration depends mainly on $R(\alpha)$. Therefore we shall choose y_i to minimize $R(\alpha)$ in a certain sense. From (2.12)

$$(3.2)\quad R(\alpha) = (\alpha - x_i)^{s+1}\{[(s+2)(\alpha - y_i) + \alpha - x_i]\, G_i(\alpha) + G_i'(\alpha)(\alpha - x_i)(\alpha - y_i).$$

As $G_i(\alpha)$ and $G_i'(\alpha)$ are in general unknown we want to minimize

$$(3.3)\quad \max(|(s+2)(\alpha - y_i) + \alpha - x_i|,\ |(\alpha - x_i)(\alpha - y_i)|).$$

One can verify that the minimal value of (3.3) is for y_i equal to y

$$(3.4)\quad y = \alpha + \frac{\alpha - x_i}{s + 2 + |\alpha - x_i|}\ .$$

As we do not know α we have to replace it by an approximation to α, z_i which depends only on the standard information, $z_i = z_i(x_i, f(x_i), \dots, f^{(s)}(x_i))$ and $z_i \neq x_i$. If $s = 0$ then $z_i = z_i(x_{i-1}, f(x_{i-1}), x_i, f(x_i))$. We define y_i as

$$(3.5) \quad y_i = z_i + \frac{z_i - x_i}{s + 2 + |z_i - x_i|} .$$

It can be proved that one can drop $|z_i - x_i|$ in the denominator without the change of the order.

Finally, y_i is defined by

$$(3.6) \quad y_i = z_i + \frac{z_i - x_i}{s + 2} .$$

Hence, from (3.6) and (2.12) we get

$$(3.7) \quad f(x) - w_i(x) = R(x)$$

for

$$R(x) = R(x,f) = (x - x_i)^{s+1}\left[(s+3)(x - z_i)\, G_i(x) + G_i'(x)(x - x_i)\frac{(s+3)(x-z_i) + x_i - x}{s + 2}\right]$$

where w_i is the interpolatory polynomial defined in Section 2.

4. THE CONVERGENCE OF THE ITERATIVE METHOD $I_{-1,s}$ FOR $s \geq 1$

In the previous section we have seen that the order of iteration mainly depends on $R(\alpha)$. From (3.7)

$$R(\alpha) = (\alpha - x_i)^{s+1}\left[(s+3)(\alpha - z_i)\, G_i(\alpha) + G_i'(\alpha)(\alpha - x_i)\frac{(s+3)(\alpha-z_i) + x_i-\alpha}{s + 2}\right]$$

Hence, to assure the maximal order of $I_{-1,s}$ for $s \geq 1$ it suffices to define approximation z_i using Newton method

$$(4.1) \quad z_i = z_i(x_i, f(x_i), \dots, f^{(s)}(x_i)) = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i = 0,1,\dots$$

<u>Theorem 1</u>

If $s \geq 1$ and

1. $f^{(s+3)}$ is a continuous function on $K(\alpha,R)$ where

$$f(\alpha) = 0 \neq f'(\alpha)$$

$$K(\alpha,R) = \{x : |x - \alpha| \leq R\}, \quad R = R(\Gamma) = \max\left(\frac{(s+3)\, C\Gamma^2 + \Gamma}{s + 2}, \Gamma\right)$$

where the constant C is defined below,

2. a real number $\Gamma > 0$ is such that

$$\Gamma h(\Gamma) < 1 \text{ and } \frac{2 M_2}{\nu_1} \Gamma < 1 \quad \text{where}$$

$$h(\Gamma) = \frac{2}{\nu_1} (2\Gamma)^s \{(1+C\Gamma) \frac{M_{s+2}}{(s+2)!} + \frac{2M_{s+3}}{(s+4)!} \frac{(s+3)(\Gamma + C\Gamma^2) + 2\Gamma}{s+2} \}$$

for

$$\nu_1 = \inf_{x \in J} \left|\frac{f(x)}{x - \alpha}\right|; \; M_i = \sup_{x \in K(\alpha,R)} |f^{(i)}(x)| \qquad i = s+2, s+3;$$

$$M_2 = \sup_{x \in J} |f''(x)|; \; C = C(\Gamma) = \frac{M_2}{2\nu_1} \frac{1}{(1 - \frac{M_2}{\nu_1} \Gamma)^2} \; ;$$

3. $x_0 \in J$ where $J = \{x : |x - \alpha| \leq \Gamma\}$,

then the sequence $\{x_i\}$ generated in $I_{-1,s}$ has the following properties:

(i) $x_i \in J, \forall i,$

(ii) $x_{i+1} - \alpha = A_i(x_i - \alpha)^{s+3}, \forall i,$

where

$|A_i| \leq A \quad \forall i$, and

$$A = \frac{2^{s+1}}{\nu_1} \{(h(\Gamma) + C) \frac{M_{s+2}}{(s+2)!} + \frac{2M_{s+3}}{(s+4)!} \cdot \frac{(s+3)(1+C\Gamma) + 2}{s+2} \}$$

(iii) $\lim_{i \to \infty} x_i = \alpha$ and

$$\lim_{i \to \infty} \frac{x_{i+1} - \alpha}{(x_i - \alpha)^{s+3}} = B \quad \text{where}$$

$$B = (-1)^{s+2} \left\{ \frac{f''(\alpha) \cdot f^{(s+2)}(\alpha)}{2(f'(\alpha))^2 \, (s+2)!} + \frac{f^{(s+3)}(\alpha)}{(s+4)!} \frac{1}{f'(\alpha)\,(s+2)} \right\}$$

■

<u>Proof of (i)</u> (By induction).

Let us assume that $x_i \in J$. From (2.12) $w_i(x) = 0$ iff $x = H(x)$ where

$$(4.2) \quad H(x) = \begin{cases} \alpha + \dfrac{1}{\frac{f(x)}{x-\alpha}} R(x) & \text{if } x \neq \alpha \\ \alpha + \dfrac{1}{f'(\alpha)} R(\alpha) & \text{if } x = \alpha \end{cases}$$

Now, using (4.2), (3.7), (2.11) and condition 2 one can verify that the assumptions of the Brouwer fixpoint theorem hold for H in the set J. Hence, there exists $x \in J$, such that $w_i(x) = 0$. So $x_{i+1} \in J$ and, by induction, $x_i \in J \quad i = 0,1,\ldots$

Proof of (ii)

From (4.2) and (3.7) we get

$$x_{i+1} - \alpha = \frac{1}{\frac{f(x_{i+1})}{x_{i+1}-\alpha}} R(x_{i+1}) \qquad i = 0,1,\ldots$$

Therefore

$$(4.3) \quad x_{i+1} - \alpha = A_i(x_i - \alpha)^{s+3} \qquad \text{where}$$

$$(4.4) \quad A_i = \frac{1}{\frac{f(x_{i+1})}{e_{i+1}}}\left(\frac{e_{i+1}}{e_i} - 1\right)^{s+1}\left\{(s+3)\left(\frac{e_{i+1}}{e_i^2} - C_i\right)G(x_{i+1}) + \right.$$

$$\left. + G'(x_{i+1})\left(\frac{e_{i+1}}{e_i} - 1\right)\frac{(s+3)\left(\frac{e_{i+1}}{e_i} - C_i e_i\right) + 1 - \frac{e_{i+1}}{e_i}}{s+2}\right\}$$

where $e_i = x_i - \alpha$, and C_i satisfies the relation

$$z_i - \alpha = C_i(x_i - \alpha)^2 .$$

An upper bound on A_i one can find using an assumption 2 and (2.11).

Proof of (iii)

From (4.3) and (i),

$$|x_{i+1} - \alpha| \le A|x_i - \alpha|^{s+3}$$

and thus

$$|x_{i+1} - \alpha| \le (A\,\Gamma^{s+2})^{i+1}|x_0 - \alpha| \quad \forall i.$$

From assumption 2 follows that $A\,\Gamma^{s+2} < 1$, and hence,

$$\lim_{i\to\infty} x_i = \alpha.$$

Finally from (4.3) and (4.4) it can be shown that

$$\lim_{i\to\infty} \frac{x_{i+1} - \alpha}{(x_i - \alpha)^{s+3}} = \lim_{i\to\infty} A_i = B,$$

which completes the proof of Theorem 1. ■

In general, B is not equal zero (see point (iii) which means that s+3 is the order of the interpolatory - integral method $I_{-1,s}$ for $s \geq 1$, (Traub [64], Wozniakowski [74]). Note that iterative methods which use only the standard information $f(x_i),\dots,f^{(s)}(x_i)$ have orders at most s+1. Additional information given by $\int_{y_i}^{x_i} f(t)dt$ increases the order of $I_{-1,s}$ by two. The usage of the $I_{-1,s}$ method in practice is profitable if the evaluation cost of the value of integral is approximately equal to the evaluation cost of function or its derivatives.

5. THE CONVERGENCE OF THE $I_{-1,0}$ METHOD

Now, we assume s = 0, which means that the information is of the form:

$$\mathfrak{N}_{-1,0} = \{f(x_i), \int_{y_i}^{x_i} f(t)\, dt\} \quad \text{where } y_i = z_i + \frac{z_i - x_i}{2} .$$

Note that we cannot now define z_i by the Newton method as we do not know the value of the first derivative. Let z_i be given now by the secant method,

$$\text{(5.1)} \quad z_i = x_i - \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} f(x_i) \quad \forall i .$$

In this case, the interpolatory - integral method $I_{-1,0}$ is a one-point method with memory (Traub [64]).

<u>Theorem 2</u>

if

1. $f^{(3)}$ is a continuous function on $K(\alpha,R) = \{x: |x - \alpha| \leq R\}$, where $f(\alpha) = 0 \neq f'(\alpha)$

$$R = R(\Gamma) = \max\left(\frac{3C\Gamma^2 + \Gamma}{2}, \Gamma\right)$$

where the constant C is defined below,

2. a real number $\Gamma > 0$ is such that

$$\Gamma \cdot h(\Gamma) < 1 \text{ and } \frac{2M_2}{\nu_1}\Gamma < 1,$$

where

$$h(\Gamma) = \frac{2}{\nu_1}\{(1 + C\Gamma) \cdot \frac{M_2}{2} + \frac{M_3}{24}[3(\Gamma + C\Gamma^2) + 2\Gamma]\}$$

for

$$\nu_1 = \inf_{x \in J}\left|\frac{f(x)}{x - \alpha}\right|, \quad M_i = \sup_{x \in K(\alpha,R)} |f^{(i)}(x)| \quad i = 2,3 ;$$

$$M_2' = \sup_{x \in J} |f''(x)|, \quad C = C(\Gamma) = \frac{M_2'}{2\nu_1} \cdot \frac{1}{\left(1 - \frac{M_2'}{\nu_1}\Gamma\right)^2}$$

3. $x_0, x_1 \in J$ where $J = \{x: |x - \alpha| \le \Gamma\}$, then the sequence $\{x_i\}$, generated in $I_{-1,0}$ has the following properties:

(i) $x_i \in J \qquad i = 0,1,\ldots$

(ii) $x_{i+1} - \alpha = A_i(x_i - \alpha)^2 (x_{i-1} - \alpha)$

where $|A_i| \le A \quad i = 0,1,\ldots$,

$$A = \frac{2}{v_1} \{(h(\Gamma) + C) \frac{M_2}{2} + \frac{M_3}{24} [3(1 + C\Gamma) + 2]\},$$

(iii) $\lim_{i\to\infty} x_i = \alpha$ and moreover

$$\lim_{i\to\infty} \frac{x_{i+1} - \alpha}{(x_i - \alpha)^2 (x_{i-1} - \alpha)} = B \quad \text{where } B = \frac{1}{4}\left(\frac{f''(\alpha)}{f'(\alpha)}\right)^2 ,$$

(iv) $\lim_{i\to\infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^p} = B^{\frac{p}{p+1}}$ where $p = 1 + \sqrt{2}$. ■

The proof of this theorem is omitted since it is similar to the proof of Theorem 1. From (iv) follows that $1 + \sqrt{2}$ is the order of the interpolatory - integral method $I_{-1,0}$.

6. THE CONVERGENCE OF THE $I_{-1,s}$ METHOD FOR MULTIPLE ZEROS

Let us now assume that $s \ge 1$ and α is an m fold zero, i.e.,

$$f(\alpha) = f'(\alpha) = \ldots\ldots = f^{(m-1)}(\alpha) = 0 \ne f^{(m)}(\alpha),$$

where $m \le s$.

The information is given by

$$\mathfrak{N}_{-1,s} = \{f(x_i), f'(x_i),\ldots,f^{(s)}(x_i), \int_{y_i}^{x_i} f(t)\,dt\} \quad \text{where}$$

$$y_i = z_i + \frac{z_i - x_i}{s + 2} \quad \text{for} \quad z_i = I_{0,s}(x_i;f)$$

The notation $I_{0,s}(x_i;f)$ is used by Traub [64] and Wozniakowski [73]. The character of convergence of the $I_{-1,s}$ method in this case is given by Theorem 3.

Theorem 3

If $s \ge 1$ and

1. $f^{(s+3)}$ is a continuous function on $K(\alpha,R) = \{x: |x - \alpha| \le R\}$ where

$$R = R(\Gamma) = \max\left(\frac{(s+3)\, D\Gamma^{\frac{s+m}{m}} + \Gamma}{s+2}, \Gamma\right)$$

where the constant D is defined below,

2. a real number $\Gamma > 0$ is such that

(i) $\Gamma^{\frac{s+1}{m} - 1} \cdot h(\Gamma) < 1,$

where

$$h(\Gamma) = \sqrt[m]{\frac{1}{\nu_m}}\, 2^{\frac{s+1}{m}} \left\{ (\Gamma + D\Gamma^{\frac{s+1}{m}}) \frac{M_{s+2}}{(s+2)!} + \frac{M_{s+3}}{(s+4)!} \cdot \frac{2\Gamma}{s+2} \left[(s+3)\, (\Gamma + D\Gamma^{\frac{s+1}{m}}) + 2\Gamma \right] \right\}^{\frac{1}{m}},$$

$$\nu_m = \inf_{x \in J} \left| \frac{f(x)}{(x-\alpha)^m} \right|, \quad M_{s+1} = \sup_{x \in J} \left| f^{(s+1)}(x) \right|$$

$$M_i = \sup_{x \in K(\alpha,R)} \left| f^{(i)}(x) \right| \qquad i = s+2,\ s+3,$$

$$D = \left(\frac{M_{s+1}}{\nu_m (s+1)!} \right)^{\frac{1}{m}} \cdot \left(\frac{1}{1 - \left(\frac{M_{s+1}}{\nu_m (s+1)!}\right)^{\frac{1}{m}} (2\Gamma)^{\frac{s+1}{m} - 1}} \right)^{\frac{s+1}{m}}$$

and

(ii) $\left(\frac{M_{s+1}}{\nu_m (s+1)!} \right)^{\frac{1}{m}} 2^{\frac{s+1}{m}} \cdot \Gamma^{\frac{s+1}{m} - 1} < 1,$

3. $x_0 \in J$, $J = \{x : |x - \alpha| \le \Gamma\}$

then

(i) $x_i \in J$, $i = 0, 1, \ldots$, $\lim_{i \to \infty} x_i = \alpha$

(ii) $|x_{i+1} - \alpha| \le A_i\, |x_i - \alpha|^{\frac{s+1+p}{m}}$ where $p = \min(\frac{s+1}{m}, 2)$.

Moreover

$A_i \le A,$

$$A = \sqrt[m]{\frac{1}{\nu_m}}\, 2^{\frac{s+1}{m}} \left\{ \Gamma^{\frac{s+1}{m} - p} (h(\Gamma) + D) \frac{M_{s+2}}{(s+2)!} + \Gamma^{2-p} \frac{M_{s+3}}{(s+4)!} \cdot \frac{2\, (s+3)(1 + D\Gamma^{\frac{s+1}{m} - 1}) + 4}{s+2} \right\}^{\frac{1}{m}},$$

(iii) $\lim_{i\to\infty} \frac{|x_{i+1} - \alpha|}{|x_i - \alpha|^{\frac{s+1+p}{m}}} \le B$ where

$$B = \sqrt[m]{\frac{m!}{|f^{(m)}(\alpha)|}} \left\{ \zeta_{\frac{s+1}{m} - p,0} \left(\frac{|f^{(s+1)}(\alpha)| \cdot m!}{|f^{(m)}(\alpha)| \cdot (s+1)!}\right)^{\frac{1}{m}} \frac{|f^{(s+2)}(\alpha)|}{(s+2)!} + \zeta_{2-p,0} \frac{|f^{(s+3)}(\alpha)|}{(s+4)!} \cdot \frac{1}{s+2} \right\}^{\frac{1}{m}},$$

$$\zeta_{i,0} = \begin{cases} 0 & \text{if} \quad i \ne 0 \\ 1 & \text{if} \quad i = 0 \end{cases} \qquad \blacksquare$$

(iv) $p(m) = (s+1+p)/m$ is the order of convergence of $I_{-1,s}$ method for m multiple zeros.

The proof of this theorem is omitted since it is similar to the proof of Theorem 1.

7. MAXIMALITY OF $I_{-1,s}$

Let $\Psi_{-1,s}$ be a class of stationary iterative methods $\varphi_{-1,s}$ which use information $\mathfrak{N}_{-1,s}$ and which have well defined order $p(\varphi_{-1,s})$ (Wozniakowski [74]). From Theorem 1 it follows that the interpolatory-integral method $I_{-1,s}$ belongs to $\Psi_{-1,s}$, $s \ge 1$.

Now we shall prove that $I_{-1,s}$ has maximal order in the class $\Psi_{-1,s}$, i.e.,

$$p(I_{-1,s}) = \sup_{\varphi_{-1,s} \in \Psi_{-1,s}} p(\varphi_{-1,s}).$$

Wozniakowski [74] defined (Definition 7) the order of information $p(\mathfrak{N})$ and proved it is equal to the maximal order. Thus, it suffices to show that in our case

$$p(I_{-1,s}) = p(\mathfrak{N}_{-1,s}).$$

<u>Theorem 4</u>

Let $\mathfrak{N}_{-1,s} = \mathfrak{N}_{-1,s}(x_i;f) = \{f(x_i),\dots,f^{(s)}(x_i), \int_{y_i}^{x_i} f(t)\,dt\}$, for any $y_i = y_i(x_i,f(x_i),\dots,f^{(s)}(x_i))$. Then

$$p(\mathfrak{N}_{-1,s}) \le s + 3.$$

If $s \ge 1$ and $y_i = z_i + \frac{z_i - x_i}{s+2}$ for $z_i = x_i - \frac{f(x_i)}{f'(x_i)}$

then $p(\mathfrak{N}_{-1,s}) = s + 3$. $\blacksquare$

<u>Proof</u>

Let $\mathfrak{F}$ be the class of complex functions of complex variable which have a simple zero and which are analytic in the neighborhood of α (see Definition 1 in Wozniakowski [74]). We recall the definition of the order of information. Let $f \in \mathfrak{F}$ and $\{f_i\} \subset \mathfrak{F}$ where

$$f(\alpha) = 0$$

(7.1) $f_i(\alpha_i) = 0, \quad i = 0,1,\ldots, \quad \lim_{i\to\infty} \alpha_i = \alpha,$

(7.2) $\lim_{i\to\infty} f_i^{(k)}(\alpha) = g^{(k)}(\alpha)$, for $k = 0,1,\ldots, g \in \mathfrak{F}$, $g(\alpha) = 0$.

Next let us assume that

(7.3) $\mathfrak{N}_{-1,s}(x_i,f) = \mathfrak{N}_{-1,s}(x_i,f_i) \quad \forall i,$

where $\{x_i\}$ is an arbitrary sequence converging to α. Let w_i be an interpolatory polynomial of degree at most $s + 1$ defined as follows:

$$\mathfrak{N}_{-1,s}(x_i,w_i) = \mathfrak{N}_{-1,s}(x_i,f), \quad \forall i.$$

Thus

$$f(x) - f_i(x) = f(x) - w_i(x) + w_i(x) - f_i(x) = R(x,f) - R(s,f_i).$$

From it and from (2.12) it follows

(7.4) $\alpha - \alpha_i = O((f(\alpha) - f_i(\alpha))) = O(|R(\alpha,f)| + |R(\alpha,f_i)|) =$

$$= O(|\alpha - x_i|^{s+1} \cdot |(s+2)(\alpha - y_i) + \alpha - x_i| + |\alpha - x_i|^{s+2} \cdot |\alpha - y_i|)$$

Moreover, we shall show that this bound is sharp. Let ζ be a number defined as follows:

$$\zeta = \begin{cases} 1 & \text{if} \quad \lim_{i\to\infty} \dfrac{\alpha - y_i}{\alpha - x_i} = -\dfrac{1}{s+2} \\ 0 & \text{otherwise} \end{cases}$$

Let

$$h_i(x) = (x - x_i)^{s+1+\zeta}(x - b_i) \quad i = 0,1,\ldots$$

where

$$b_i = \frac{(s+2+\zeta)\, y_i + x_i}{s + 3 + \zeta}.$$

Setting

(7.5) $f_i(x) = f(x) + h_i(x), \quad \forall i$

for any function $f \in \mathfrak{F}$, $(f(\alpha) = 0,\ f_i(\alpha_i) = 0)$ one can verify that conditions (7.1), (7.2) and (7.3) hold. Next, there exist constants C_i 0 such that $\lim_{i\to\infty} C_i = C$ 0 and

$$(7.6)\quad |\alpha - \alpha_i| = C_i |f(\alpha) - f_i(\alpha)| = C_i |\alpha - x_i|^{s+1+\zeta} |\alpha - b_i| = \frac{C_i}{s+3+\zeta} |\alpha - x_i|^{s+1+\zeta} |(s+2+\zeta)(\alpha - y_i) + \alpha - x_i|$$

which proves that (7.4) is sharp.

From (7.4) and (7.6) it follows that for any y_i the order of information $p = p(\mathfrak{N}_{-1,s})$ exists. Let us assume that $p > s+3$. Let $\epsilon > 0$ be a number such that $p - \epsilon > s + 3 + \epsilon$. For f and $\{f_i\}$ given by (7.5) we get from (7.6):

$$+\infty \geq \limsup_{i\to\infty} \frac{|\alpha - \alpha_i|}{|x_i - \alpha|^{p-\epsilon}} \geq \limsup_{i\to\infty} \frac{|\alpha - \alpha_i|}{|x_i - \alpha|^{s+3+\epsilon}} = +\infty$$

which is a contradiction. Hence $p \leq s+3$ for any $y_i = y_i(x_i, f(x_i) \ldots f^{(s)}(x_i))$. Now we shall show that the above estimation of p is achievable for $s \geq 1$.

Indeed, setting

$$y_i = z_i + \frac{z_i - x_i}{s+2}, \quad z_i = x_i - \frac{f(x_i)}{f'(x_i)}$$

we have

$$\lim_{i\to\infty} \frac{\alpha - y_i}{\alpha - x_i} = -\frac{1}{s+2} \quad (\text{so } \zeta = 1).$$

It is easy to verify that from (7.4) and (7.6) it follows

$$p(\mathfrak{N}_{-1,s}) = s + 3,$$

which completes the proof of Theorem 3. ■

From Theorems 1 and 3 we get

Corollary 1

The interpolatory - integral method $I_{-1,s}$ is maximal, i.e.,

$$p(I_{-1,s}) = p(\mathfrak{N}_{-1,s}). \quad ■$$

References

Traub [64] J. F. Traub, Iterative Methods for Solutions of Equations, Prentice-Hall, 1964.

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